THE CREATION OF THE THEORY OF GROUP CHARACTERS

by Thomas Hawkins

The creation of the theory of group characters and representations by Georg Frobenius is one of those relatively rare episodes in the history of mathematics for which we have much more information at our disposal than that conveyed through the published papers themselves. Considerable insight is provided by the correspondence between Frobenius and Richard Dedekind during the years 1895-1898. Portions of Dedekind's letters were published by Emmy Noether in the second volume of his collected works (1931). But Frobenius's share of the correspondence appeared to be lost to posterity until, about six years ago, it was located by chance in Philadelphia. Among the 178 pages penned by Frobenius are many that are especially important because they reveal the stages by which he arrived at the theory of characters. In what follows I shall describe some of the less technical aspects of the origins and creation of the theory as they are revealed through the Dedekind-Frobenius correspondence.[1]

Nowadays a character on a finite group is defined as the trace of a matrix representation, and the properties of characters are derived from those of the representations. Originally characters were introduced in a different manner. The underlying concept was that of a group determinant, not a group representation. It will be helpful to begin by defining this determinant and relating its properties to those of matrix representations.

Let $\mathfrak{H}$ denote a finite group of order h with elements E (the identity), A, B, ... Each element has associated with it an independent variable: x_E, x_A, x_B, ... The group determinant $\Theta = \Theta\,(x_E, x_A, x_B, \ldots)$ is the homogeneous polynomial of degree h defined by

$$(1) \qquad \Theta = \begin{vmatrix} x_{EE^{-1}} & x_{EA^{-1}} & x_{EB^{-1}} & \ldots \\ x_{AE^{-1}} & x_{AA^{-1}} & x_{AB^{-1}} & \ldots \\ x_{BE^{-1}} & x_{BA^{-1}} & x_{BB^{-1}} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} .$$

If $\varrho: H \to \varrho(H)$ denotes the regular representations of $\mathfrak{H}$, then

$$\Theta = \det [x_E\varrho(E) + x_A\varrho(A) + x_B\varrho(B) + \ldots] .$$

Furthermore if M is a matrix of constants such that

$$M^{-1}\varrho(H)M = \begin{pmatrix} \mu(H) & O \\ O & \nu(H) \end{pmatrix} \quad \forall H \in \mathfrak{H} ,$$

then the decomposition of the representation corresponds to a factorization of the group determinant:

$$\Theta = \Phi(x_E, x_A, x_B, \ldots)\Psi(x_E, x_A, x_B, \ldots)$$

where

$$\Phi = \det [\sum_H x_H\mu(H)] , \quad \Psi = \det [\sum_H x_H\nu(H)] .$$

Note in particular that the degrees of the polynomials Θ, Φ, Ψ equal the degrees of the corresponding representations ϱ, μ, ν.

The theory of group characters and representations was developed in response to the following problem. Since Θ is a homogeneous polynomial of degree h, it can be factored into distinct irreducible homogeneous polynomials Φ_i with complex coefficients:

$$(2) \qquad \Theta = \prod_{i=1}^{\ell} (\Phi_i)^{e_i} , \qquad f_i = \text{degree } \Phi_i .$$

The irreducible factors are uniquely determined up to a constant factor. To specify them completely, we require that the coefficient of $x_E^{f_i}$ in Φ_i be 1. This is permissible because the coefficient of x_E^h in Θ is 1. Now Θ is determined by the nature of the group $\mathfrak{H}$, as (1) clearly indicates. The problem is therefore to determine how the factorization (2) reflects the structure of the underlying group $\mathfrak{H}$. For example, how do the irreducible factors Φ_i, their number ℓ, their degrees f_i, and their multiplicities e_i relate to $\mathfrak{H}$?

This problem was studied by Frobenius in 1896 and led to his creation of the theory of group characters. At first glance, the problem seems to be just the type one would expect to arise within the context of nineteenth-century mathematics, for the theory of groups and the theory of determinants were characteristic products of that century. But the problem depends for its formulation upon the concept of the group determinant, and that concept was as unfamiliar in the nineteenth century as it is today. How, then, was the concept introduced, how did Frobenius become interested in it and how did it involve the theory of characters? This is the question I shall now consider. As we shall see, Dedekind figures prominently in the answer.

Dedekind had received most of his mathematical education while at the University of Göttingen, where he studied mathematics at the pre- and postdoctoral levels from 1850 to 1858. His teachers included Gauss and Dirichlet, who succeeded Gauss in 1855. Both had made fundamental contributions to the theory of numbers, and it was upon this area of mathematics that Dedekind decided to concentrate his own efforts. He perfected the work of his teachers and complemented it with his own innovations, such as his theory of ideals in algebraic number fields. One way in which he perfected their work was through the introduction of the concept of a character on an abelian group.

I cannot enter into any of the details of the matter here.[2] Let me simply point out that many arithmetical problems studied by Gauss and Dirichlet involved what we now recognize as abelian groups: groups of equivalence classes of binary quadratic forms of fixed determinants under Gaussian composition, and groups of residue classes of integers relatively prime to a fixed integer. Viewed in retrospect, their work can also be said to have involved group characters. Gauss spoke of the character of a form class to designate certain characteristic properties of the class that are reflected in the nature of the integers representable by its members. Gauss's characters can be used to specify characters in the modern sense. Dirichlet had introduced expressions χ defined on form classes which take the values ± 1, but he did not term them characters or relate them to Gaussian characters. Also Dirichlet's work with L-series involved products of roots of unity which were associated with residue classes of integers. It was Dedekind, however, who first perceived and appreciated the unity that could be achieved by defining a character on an abelian group $\mathfrak{H}$ as any complex-valued function $\chi \not\equiv 0$ such that

$$\chi(AB) = \chi(A)\chi(B) \qquad \forall\, A,B \in \mathfrak{H} .$$

Dedekind first presented this definition in the 1879 edition of Dirichlet's lectures on the theory of numbers.

Dedekind spent all but the early years of his career in his native city of Brunswick, where he lived with his sister and taught at the local technical institute. He was a timid person who preferred the familiarity and security of his home town to the competition of a more distinguished university. As in the case of Gauss, his publications failed to reveal the actual scope of his mathematical investigations. Many interesting questions were suggested to him by his work on the theory of algebraic numbers, questions which were tangential to his primary research interests and which he therefore tended to drop after pursuing them a while. Two such questions had considerable importance in the creation of the theory of group characters for non-abelian groups. Although they were unrelated, the fact that Dedekind considered them together in February 1886 turns out to be significant.

The first question is as follows: The theory of algebraic numbers involves consideration of finite normal extensions of the field of rational numbers.

The subfields of such an extension need not be normal, and Dedekind posed the problem of determining the nature of those extensions for which all subfields are, in fact, normal. For such an extension the associated Galois group would have the property that all its subgroups are normal. This suggests a purely group-theoretic question: characterize those non-abelian groups for which all subgroups are normal. Whether or not Dedekind thought of the purely group-theoretic question in 1886 is not clear, but we shall see that the related question for fields prompted him to consider it later.

The second question that Dedekind investigated in February 1886 was actually an extension of a question he had considered about 1880, the year after he introduced the concept of a character on an abelian group. Let K denote a finite normal extension of the field of rational numbers and $\mathfrak{H} = \{\pi_1, \pi_2, \ldots, \pi_n\}$ the Galois group of K. The discriminant of $w_1, w_2, \ldots, w_n \in K$ is defined as $\Delta = D^2$, where $D = \det(w_j\pi_i)$. In particular, if $w_j = w\pi_j$ then $D = \det(w\pi_j\pi_i)$. Simply by analogy, Dedekind was led to consider $\det(x_{\pi_j\pi_i})$ and then $\det(x_{\pi_j\pi_i^{-1}})$. It was known at this time that a determinant with rows that are cyclical permutations of the first can be factored into linear factors with roots of unity as coefficients. For example

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix} = (a_1 + a_2 + a_3)(a_1 + wa_2 + w^2a_3)(a_1 + w^2a_2 + wa_3)$$

where w is a primitive cube root of unity. Dedekind discovered that if $\mathfrak{H}$ is any abelian group, then the associated group determinant always factors into linear factors with coefficients given by the characters of $\mathfrak{H}$. That is, if $\mathfrak{H}$ is abelian, then

$$\Theta = \prod_{s=1}^{h} \left(\sum_{H} \chi_s(H) x_H \right) , \tag{3}$$

where $\chi_1, \chi_2, \ldots, \chi_h$ denote the characters of $\mathfrak{H}$.

In 1886 Dedekind returned to the investigation of group determinants and considered the group determinant of non-abelian groups. To see how they behave, he computed some examples. First he considered $\mathfrak{H} = S_3$ and obtained (by a change of variables) the factorization

$$\Theta = (u + v)\ (u - v)\ (u_1u_2 - v_1v_2)^2 \ . \tag{4}$$

Because of the irreducible second-degree factor in (4), the group determinant of S_3 cannot be factored into linear factors with complex coefficients. Now at this time Dedekind had just composed some papers on commutative hypercomplex number systems (linear associative algebras over the complex field). Perhaps this work prompted him to explore the possibilities of factoring Θ completely into linear factors by using hypercomplex numbers as coefficients.

For S_3 Dedekind did devise a hypercomplex number system—the group albegra of S_3 modulo an ideal—which made it possible to split $u_1u_2 - v_1v_2$ into linear factors, but it was the next example that excited and encouraged him. He considered the quaternion group of order 8 and obtained (again by means of a variable change) the following factorization for Θ:

$$(5)\ (u_1+u_2+u_3+u_4)\ (u_1+u_2-u_3-u_4)\ (u_1-u_2+u_3-u_4)\ \cdot \\ (u_1-u_2+u_3+u_4)\ (v_1^2+v_2^2+v_3^2+v_4^2)^2 .$$

What impressed him was that the irreducible second-degree factor can be interpreted as the norm of a quaternion:

$$v_1^2 + v_2^2 + v_3^2 + v_4^2 = (v_1 + iv_2 + jv_3 + kv_4)\ (v_1 - iv_2 - jv_3 - kv_4) .$$

Thus the group determinant of the quaternion group factors completely into linear factors if quaternions are permitted as coefficients.

As these examples indicate, Dedekind's interest in the group determinant was focused upon the problem of extending (3) to non-abelian groups by determining for a given group $\mathfrak{H}$ a hypercomplex number system over which the group determinant Θ would factor into linear terms. Presumably he hoped to find some interesting relations between the structure of $\mathfrak{H}$ and the hypercomplex number system. Dedekind's line of approach to the study of group determinants never appealed to Frobenius, to whom we must now turn. Frobenius ended up studying the relation between $\mathfrak{H}$ and Θ by extending the concept of a character rather than the domain of coefficients admissible for its factorization. I say "ended up" because, as we shall see, he did not set out consciously to generalize the concept of a character.

Frobenius was eighteen years Dedekind's junior. He had written his doctoral thesis at Berlin in 1870, under Weierstrass's direction, on series solutions to differential equations. After holding various teaching positions in Berlin, he was appointed to a professorship at the Polytechnikum in Zürich (now the E.T.H.) in 1875. At the Polytechnikum, Frobenius produced a steady stream of papers on diverse mathematical subjects. Whether the subject was differential equations or theta functions, however, it was the algebraic aspects of the theory that attracted his attention. He was especially fond of what was known in Berlin as the theory of forms—the theory of quadratic and bilinear forms—which would now be called the theory of matrices or linear algebra.[3] By the early 1890s a new subject had begun to fascinate him: the theory of abstract finite groups.[4]

Frobenius's new interest was pursued in new surroundings, for in 1893 he succeeded Kronecker (who died in 1892) as Professor at the University of Berlin. The professorship at Berlin brought with it membership in the Berlin Academy of Sciences. In his inaugural address before the Academy, Frobenius described his research interests. After admitting that algebraic questions were his first love, he explained further that "both directions in modern algebra, the theory of equations and the theory of forms, especially captivated me. In the latter I was drawn by preference to the theory of determinants, in the former to the theory of groups." Little did he realize that he would soon be in a position to combine his two favorite subjects, the theories of groups and determinants, in his study of the group determinant.

In 1893, when Frobenius made the above statement, he had never heard of group determinants. He had known Dedekind since at least 1880 when the latter visited Zürich and was entertained at Frobenius's home. It was a pleasant occasion for all concerned, and from that time on Frobenius and Dedekind began to correspond sporadically on questions relating to the theory of numbers—Dedekind's primary interest. The subject of group determinants was never mentioned until 1895. It happened as follows.

On January 19, 1895, Dedekind wrote a letter to Frobenius which was apparently the first since Frobenius had moved to Berlin. Dedekind had decided that a letter he had received from Kronecker in 1880 was significant enough to warrant publication. Since Frobenius had written the memorial address on Kronecker and his contributions to mathematics, it was only natural that Dedekind should send Kronecker's letter to him with the suggestion that it appear in the proceedings of the Berlin Academy.

Frobenius responded with a long, friendly letter. Besides expressing his agreement with Dedekind's suggestion, he touched on many matters of common interest: a quarrel between Dedekind and Hilbert, Weierstrass's failing health, the reactions of the Frobenius family to their new surroundings, and so on. One passing remark turned out to be consequential. Frobenius wrote: "I am curious what you will say about my work on the theory of groups that I will present next to the Academy. I mean, I know you have concerned yourself with the subject, but I do not know how far you have gone into it."[5] Dedekind answered as follows:

> I am very excited about your work on groups since I was pleased with the simplicity of your methods, among others your proof that in a group whose order is divisible by the prime number p there is always an element of order p. In the first years of my studies on groups (1855-1858) I arrived at it in a much more involved way. Later I pursued certain questions about groups only insofar as the motivation arose from other quarters: therefore, if it should happen that I at some point already considered the subject of your work, I would certainly not have advanced as far as you. For good measure, let me ask: do hypercomplex numbers with noncommutative multiplication also intrude in your research? But I do not wish to trouble you for an answer, which I will best obtain from your work.[6]

What was Frobenius's response to the cryptic allusion to hypercomplex numbers and groups? Before the letters of Frobenius were discovered, I assumed that he must have been sufficiently curious to request a clarification from Dedekind, for non-commutative hypercomplex numbers were not customarily considered in relation to groups. But Frobenius expressed no curiosity whatsoever in his answer: "My work on groups is now appearing. There is no discussion of hypercomplex numbers. Previously obtained results are summarized, the methods of Sylow are further developed, and the investigations in my last work are carried further."[7] Faced with Frobenius's indifference, Dedekind replied a bit apologetically that his "question regarding the use of hypercomplex numbers in the theory of groups was very audacious. It arose from an observation I made in February 1886 but then did not pursue further, although it seemed noteworthy enough to me. Perhaps sometime I will venture to present it to you at the risk that it will entirely vanish before your criticism. . . ."[8]

The letters that passed between Dedekind and Frobenius in January and February of 1895 were occasioned by the business of getting Kronecker's letter of 1880 published in an expurgated form acceptable to Dedekind. Dedekind's letter of February 12 finished that business, and so there was no reason to continue the correspondence—unless, of course, Frobenius was now sufficiently curious about Dedekind's use of hypercomplex numbers in the theory of groups to write and encourage him to present his ideas. Frobenius was not, however, and so the correspondence broke off. Had Dedekind never returned to the matter, Frobenius probably would not be known as creator of the theory of group characters and representations. This does not mean that the theory would not have been created during this period.

Although group determinants appear to have been Dedekind's private property, an entirely different approach to the theory was at hand in the 1890s, namely the approach through the group algebra and its structure. As a matter of fact, some of the basic properties of matrix representations and characters were discovered in this manner by Theodor Molien (1897); and Burnside's study (1898) of Lie groups defined by finite groups was also leading in the same general direction.[9] But if Frobenius had never learned about group determinants from Dedekind it is very unlikely he would have followed a route similar to that taken by Molien or Burnside. Their work was motivated by Lie's theory of transformation groups, and Frobenius had an extremely low opinion of Lie and his theory.[10]

Frobenius consequently was fortunate that Dedekind decided to renew their correspondence, after a lapse of one year, and to tell him about group determinants. During the fall of 1895 Dedekind decided to pursue some research of his own on groups, perhaps stimulated by Frobenius's inquiry as to whether he had done any work in this area. We saw that in February 1886 Dedekind had considered normal extensions of the rational number field with the property that all subfields are normal. In 1895 he now considered the related group-theoretic problem: characterize those non-abelian groups with the property that all subgroups are normal—Hamiltonian groups as he now called them. To his surprise, he discovered the answer was relatively simple, and he communicated it to his close friend Heinrich Weber. Weber was an editor of *Mathematische Annalen* and urged his friend to publish his result there.

Dedekind, however, did not believe in rushing into print. He wanted to be certain the result was new. Perhaps he also wanted to make certain it was significant—that is, not a simple consequence of known results. He was not well-versed in the current literature on groups, whereas Frobenius was. Dedekind therefore wrote to him on March 19, 1896:

> Some time ago I had intended to write to you and to first of all express my thanks for your works, through which you so successfully brighten the African darkness of the theory of groups. I also wanted to communicate some studies on groups and fields which, in Weber's opinion, contain new results but which do not touch upon the same areas as yours. I am, however, reluctant to engage you in conversation right now and prefer to wait until you feel more inclined towards it.

The reason for Dedekind's hesitancy was that he heard from Frobenius's younger colleague at Berlin, Kurt Hensel, that Frobenius was not feeling well. Frobenius wrote back assuring Dedekind that he was well enough to discuss mathematics and invited him to communicate his discoveries.

Dedekind accepted the invitation and communicated his theorem on Hamiltonian groups in a letter of March 25. After presenting it, he added: "Since I am speaking about groups, I would like to mention another consideration which I came upon in February [1886]." He then proceeded to define the group determinant, to state the theorem (3) about its factorization for abelian groups, and to suggest a link between the number of linear factors in the non-abelian case and normal subgroups $\mathfrak{U}$ of $\mathfrak{H}$ such that $\mathfrak{U}_{RS} = \mathfrak{U}_{SR}$.

Frobenius immediately and enthusiastically responded with an eighteen-page letter (March 29, 1896):

> Long ago it surprised me that you had not participated more actively in the development of the theory of abstract groups, even though, by virtue of your disposition, this field must have been especially attractive to you. Now I see that you have concerned yourself with it for ten years and have kept back your extremely beautiful results from your friends and admirers (also, unfortunately, by virtue of your disposition?).

Most of the eighteen pages are filled with a technical discussion of Dedekind's theorem on Hamiltonian groups, but Frobenius commenced his letter with some remarks about group determinants, a subject that obviously interested him: "I believe I am fairly knowledgeable about the theory of determinants, and I think that the formula [i.e., (3)] . . . has not been expressed in this generality for abelian groups. For cyclic groups it has been known a long time. . . . But I also never thought of this generalization which is so close at hand." In his letter, Dedekind had briefly indicated the significance of hypercomplex numbers for the factorization of the group determinant into linear factors. Frobenius did not find this approach appealing: "I do not know yet whether I will be able to reconcile myself to your hypercomplex numbers." From the outset he was more interested in the ordinary factorization of Θ and its relation to $\mathfrak{H}$: "the entire subject is so new to me that I cannot yet see how the irreducible factors of the determinant are connected with the (invariant [= normal]?) subgroups. If you know something more about this, please tell me."

In response to the request for further information, Dedekind conjectured the following theorem:[11]

> *Theorem A. The number of linear factors in the factorization of* Θ *over the complex numbers is equal to the index of the commutator subgroup* $\mathfrak{H}'$ *and hence to the order of the abelian group* $\mathfrak{H}/\mathfrak{H}'$.

He even suggested a line of proof by remarking that the linear factors correspond "in a certain manner" to the characters of the abelian group $\mathfrak{H}/\mathfrak{H}'$. Finally, he invited Frobenius to investigate these matters since "I distinctly feel that I will not achieve anything here." He undoubtedly felt as well that the study of Θ would take him too far afield from his principal interests in the theory of numbers. Time and energy should not be squandered at age sixty-five. The study of Θ was nevertheless a good research problem, one well suited to Frobenius's tastes, and he clearly wanted Frobenius to pursue it.

Dedekind's irresistible invitation arrived during the break between the winter and summer semesters when Frobenius had more time for research. His next letter (April 12, 1896) indicates that he was spending that time probing into the mysteries of Θ from every conceivable angle. Dedekind received the letter nine days after he had sent off the invitation and it was twenty-four pages long! It is an extraordinary document because it clearly was written in stages that reflect Frobenius's progress in seeking to unlock the mysteries of the group determinant.

The first matter considered by Frobenius was, of course, Dedekind's conjectured Theorem A. Following Dedekind's hint, he showed that every character on the abelian group $\mathfrak{H}/\mathfrak{H}'$ defines in the obvious way a function χ on $\mathfrak{H}$ which is a character in Dedekind's sense: $\chi(AB) = \chi(A)\chi(B)$. The character χ determines a linear factor of Θ, namely $\sum_H \chi(H)x_H$; conversely, every linear factor is of this form. At this stage, Frobenius conceived of characters as had Dedekind, although he considered them for non-abelian groups since the above proof required such a context.

The next line of investigation pursued by Frobenius is especially significant, for it involved the introduction of the functions on $\mathfrak{H}$ which are the characters of the irreducible representations of $\mathfrak{H}$, although initially he did not regard them as generalized characters nor fully appreciate their importance. Indeed, after he had disposed of Dedekind's conjectured theorem, Frobenius was unsure how to proceed next: "Naturally all the irreducible factors . . . of $[\Theta]$ . . . , and the powers to which they occur, must derive from the group. . . . However, I still have no idea how." In order to gain some insight, he tried various lines of attack. One of his strategies was to try to transfer properties of Θ to the irreducible factors Φ. Since Θ is a determinant, many of its properties could be obtained from well-known properties of determinants.

One such property is

$$A \cdot \operatorname{adj} A = (\det A) I \tag{6}$$

where adj A is the adjoint matrix corresponding to $A = (a_{ij})$. Since the (i,j) entry in adj A is $\partial D/\partial a_{ij}$, where $D = \det A$, (6) was frequently expressed in the following manner:

$$\sum_j a_{ij} \frac{\partial D}{\partial a_{jk}} = \delta_{ik} . \tag{7}$$

Specialized to the group determinant, (7) (with $i \neq k$) can be expressed in the form:

$$\sum_R \frac{\partial \Theta}{\partial x_R} x_{AR} = 0 , \quad A \neq E . \tag{8}$$

To transfer property (8) to the irreducible factors, Frobenius simply wrote $\Theta = \prod_{i=1}^{\ell} (\Phi_i)^{e_i}$ and applied the product rule for derivatives. The resulting equation, after some straightforward manipulations, becomes

$$\sum_i e_i \Theta_i \left\{ \sum_R \frac{\partial \Phi_i}{\partial x_R} x_{AR} \right\} = 0 \quad (A \neq E) , \tag{9}$$

where $\Theta_i = \prod_{j \neq i} \Phi_j$.

Now since Φ_k and Θ_k are relatively prime whereas Φ_k divides Θ_i for $i \neq k$, (9) implies that Φ_k must divide $\Delta_k = \sum_R \frac{\partial \Phi_k}{\partial x_R} x_{AR}$. This must be true in particular if (dropping the subscript k) Φ and Δ are regarded as polynomials in x_E. If we write

$$\Phi = x_E^f + x_E^{f-1} \left(\sum_{A \neq E} \psi(A) x_A \right) + \ldots ,$$

so that $\psi(A)$ simply denotes the coefficient of $x_E^{f-1} x_A$, then direct calculation shows that

$$\Delta = \psi(A^{-1}) x_E^f + \ldots .$$

Since Φ divides Δ we therefore obtain

$$\sum_R \frac{\partial \Phi}{\partial x_R} x_{AR} = \psi(A^{-1})\Phi \; . \tag{10}$$

If we define $\psi(E) = f$, then (10) is true for $A = E$ as well.

Equation (10) thus represents the property of the irreducible factor Φ corresponding to the property (8) of Θ. It turns out that the function ψ introduced in this process is the character of the irreducible representation of $\mathfrak{H}$ corresponding to Φ in the manner indicated in my introductory remarks. When Φ is linear, Frobenius's proof of Dedekind's Theorem A shows that $\Phi = \Sigma\, \chi(A)x_A$, where χ is a character in Dedekind's sense. Thus when Φ is linear $\psi = \chi$: the functions ψ include the Dedekind characters. Of course Frobenius had not purposely set out to generalize the concept of a character and nowhere in his letter did he regard the functions ψ as generalized characters. But as his investigation proceeded, as he obtained further relationships by a variety of strategies, the importance of (10) and of these functions became increasingly apparent.

By the end of the letter Frobenius realized that all the coefficients of Φ are rationally expressible in terms of the values of ψ. He had also discovered the orthogonality relation

$$\sum_R \psi_i(AR^{-1})\psi_j(RB^{-1}) = \frac{h}{e_i}\psi_i(AB^{-1})\delta_{ij} \tag{11}$$

and used it to prove the following two theorems suggested by Dedekind's examples. (See (4) and (5).)

Theorem B. A linear change of variables can be made so that each Φ_i *becomes a function of a distinct set of* v_i *independent variables.*

Theorem C. $\mathrm{v}_i = \mathrm{e}_i\mathrm{f}_i$.

In spite of these discoveries, Frobenius was not very cheerful when he concluded his letter. He had not discovered any theorems like Dedekind's Theorem A which related the structure of Θ (the number of linear factors) to that of $\mathfrak{H}$ (the number of cosets relative to $\mathfrak{H}'$). Frobenius's main results were Theorems B and C, which were not as satisfying. The proofs were also long and extremely complicated, and Frobenius confessed to Dedekind that his proof of Theorem C was "so complicated that I myself do not rightly know where the mainpoint* of the proof is." The asterisk refers to a footnote added by Frobenius as an afterthought. Perhaps, he suggested, the mainpoint might be contained in the relation

$$\psi(AB) = \psi(BA) \; , \tag{12}$$

which he had used in the proof to establish that the matrices $(x_{AB^{-1}})$ and $(\psi(AB^{-1}))$ commute. And so he closed his letter meditating upon the significance of (12).

Five days later (April 17, 1896) he wrote in jubilation to Dedekind:

> My former colleague Schottky[12] was and is one of the greatest optimists that I know; otherwise he would not have been able to endure my pessimism so well. He used to say: If in an investigation, after difficult mental exertion, the feeling arises that nothing will be achieved on the matter in question, then one can rejoice for he is standing before the solution. Many times I have found this truth confirmed and this time as well. At the end of my last letter I gave up the search and demanded your assistance. The next day I saw, if not the entire solution, at least the way to it. My feeling that the equation $\psi(AB) = \psi(BA)$ provided the key did not deceive me. I still have a long way to go but I am certain I have chosen the right path. . . . Do you know of a good name for the function ψ? . . . Or should ψ be called the character of Φ (which agrees for linear Φ)?

How had the relation (12) opened up new perspectives for Frobenius? He realized that (12) holds for a function if and only if it is constant on the conjugate classes of the group. The functions ψ thus being "class functions," Frobenius decided to express them and their properties in a notation that reflected this fact. When the orthogonality relation (11) is expressed in the new notation, it becomes clear that the number (ℓ) of distinct irreducible factors Φ_i of Θ is less than or equal to the number of conjugate classes of the group. Frobenius naturally hoped to be able to prove that these two numbers are equal, as they are in Dedekind's examples. He succeeded by deriving the second orthogonality relation (summation over the irreducible characters). The result was especially pleasing to him because, like Dedekind's Theorem A, it related the factorization of Θ directly to the structure of the group. Encouraged by this result, he went on to make further discoveries.

By the end of his second "progress report" to Dedekind, Frobenius had obtained most of the basic theorems of the theory of group characters and representations—expressed of course in the language of group determinants. But there was one notable exception. The factorization of Θ for the symmetric group S_3 and the quaternion group (equations (4) and (5)) and for the dihedral group D_4, which Frobenius worked out, suggested that the multiplicity of each irreducible factor is equal to its degree—that is, that (in the notation of (2)) e_i is always equal to f_i. In the language of matrix representations, the presumed theorem is that each irreducible representation occurs in the regular representation as often as its degree. Frobenius finally proved the theorem, but it took him over five months of effort and caused him considerable anguish. I will conclude by indicating some of the nontechnical aspects of the manner in which he finally obtained a proof.

Frobenius's proof of Theorem A showed that when $f = 1$, $e = 1$. Before long he managed to prove that if $f = 2$ then $e = 2$. The proof unfortunately would not generalize to higher values of f. The three computed examples also provided no evidence of what occurs for $f > 2$ since all factors are of first or second degree. Frobenius clearly wanted the equality to hold in general. As he wrote to Dedekind on April 26, 1896: "It would be wonderful if $e = f$. For then my theory would supply everything needful to determine the prime factors Φ."

To assure himself that his expectation of the validity of the theorem was justified, he decided to work out some more examples—examples of groups of higher order which were more likely to have irreducible factors of degrees larger than 2. He was an avid calculator and, despite the magnitude of the calculations involved, he studied five examples: projective unimodular groups of orders 12, 60, and 168 and the symmetric groups S_4 and S_5 of orders 24 and 120. For the group of order 12 (the tetrahedral group), he actually calculated and factored the group determinant. It has an irreducible factor of degree $f = 3$ which occurs to the third power so that $e = f$. In view of the size of the group determinant in the remaining examples, Frobenius did not attempt to factor them. He contented himself with the computation of the numbers $g = ef$. They can be represented as the characteristic roots of a k-by-k matrix, where k is the number of conjugate classes of the group. The value of k ranges between 4 and 7 in the above examples so that the computations involved are manageable. In each case the numbers $g = ef$ turned out to be squares, thereby lending further support to the hypothesis that $e = f$. But Frobenius was still unable to prove it and so he wrote in exasperation to Dedekind: "Should you have an example where $e \neq f$, please write to me as soon as possible so that I will not go astray."[13]

Dedekind had no counter-example to offer, but Frobenius still had no proof. After a month of silence, Frobenius wrote to Dedekind in a happier mood. He thought (incorrectly) that he was on the verge of a proof that $e = f$, and he decided to share the secret of his success with his friend:

> I quickly realized I would not attain the goal [of proving $e = f$] with my puny means and I decided to seek the "great mean." I call it the "Principle of the Horse Trade." You . . . know how a horse is bought (or a diamond or a house). At the market, the desired horse is ignored as much as possible and at last is allowed to be formally recognized.
>
> It can also be called, in more elegant language, the "Principle of the Pout." Therefore, in order to find $e = f$, I first of all went to the trade exhibition with my wife, then to the art exhibition. At home I read *Effie Briest*[14] and rid my fruit trees of caterpillars. . . .
>
> I gather from many places in your writings that my "Method of the Horse Trade" is probably known to you, albeit by a more civilized name. I hope you will not give away the trade secret to anyone. My great work *On the Methods of Mathematical Research* (with an appendix on caterpillar-catching), which makes use of it, will appear after my death.[15]

Little did Frobenius realize that, in a somewhat unexpected way, his

"Principle of the Horse Trade" would provide him with the proof that $e = f$. Adhering to the Principle, he proceeded to ignore the problem of finding a proof and busied himself with other matters. He published his first two papers on group characters (presented in such a manner that the connection with Θ and Φ is obscured), meditated on the possible significance of Dedekind's hypercomplex factorization of Θ (which he continued to find unappealing), and published some old investigations on the theory of numbers. Then, the summer semester having ended, he travelled to Juist, one of the East Frisian Islands, for a vacation.

When Frobenius returned to Berlin and turned once again to the problem of proving that $e = f$, he found to his dismay that, as a result of the time lapse and the disorderly state of his papers he could no longer find, or recall, his latest proof that $f = 2$ implies $e = 2$—the proof he once believed he was on the verge of extending to $f > 2$. "After much torment," he finally devised another proof that $f = 2$ implies $e = 2$. Much to his delight he discovered that this proof could be generalized. At long last he had proved that $e = f$ and could now publish his paper on the group determinant!

Postscript

At Dedekind's urging Frobenius translated his results on the group determinant into the language of matrix representations. In the process he discovered that his generalized characters ψ_i are the trace functions of the irreducible representations of the group.

NOTES

1. For further details consult T. Hawkins, "New Light on Frobenius' Creation of the Theory of Group Characters," *Archive for History of Exact Sciences* **12** (1974): 217-243.

2. They are given in section 2 of T. Hawkins, "The Origins of the Theory of Group Characters," *Archive for History of Exact Sciences* **7** (1971): 142-170.

3. Frobenius and Weierstrass figure much more prominently in the history of the theory of matrices than is generally realized. See T. Hawkins, "Another Look at Cayley and the Theory of Matrices," *Archives Internationales d'Histoire des Sciences* **26** (1977): 82-112; "Weierstrass and the Theory of Matrices," *Archive for History of Exact Sciences* **17** (1977): 119-163. The significance of Frobenius's contributions is discussed in sections 5 and 6, respectively.

4. On the status of the theory in 1890, consult H. Wussing, *Die Genesis des abstrakten Gruppenbegriffes* (Berlin: VEB Deutscher Verlag der Wissenschaften, 1969).

5. Letter dated January 24, 1895. Frobenius's letters to Dedekind and drafts of Dedekind's letters to Frobenius are located in the archives of the Clifford Memorial Library, University of Evansville, Indiana. All translations of the Dedekind-Frobenius correspondence are my own.

6. Letter dated February 8, 1895: R. Dedekind, *Gesammelte mathematische Werke,* vol. 2, pp. 419-420.

7. February 10, 1895.

8. February 12, 1895: *Werke*, vol. 2, p. 420.

9. The work of Molien and Burnside is examined in T. Hawkins, "Hypercomplex Numbers, Lie Groups and the Creation of Group Representation Theory," *Archive for History of Exact Sciences* **8** (1972): 243-287.

10. K.-R. Biermann, *Die Mathematik und ihrer Dozenten an der Berliner Universität 1810-1920* (Berlin: Akademie Verlag, 1973), p. 216.

11. Dedekind, *Werke*, vol. 2, p. 422.

12. Friedrich Hermann Schottky had also done his doctoral dissertation at Berlin under Weierstrass (1875). He was a professor at the University of Zürich (1882-1892) when Frobenius was at the nearby Polytechnikum. Schottky became Frobenius's colleague once again when he joined the faculty of the University of Berlin in 1902.

13. April 26, 1896.

14. A novel by Theodor Fontane published in 1895.

15. June 4, 1896.

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